UNIVERSITY OF ILLINOIS
DIGITAL COMPUTER

AUXILIARY
LIBRARY ROUTINE V 14 - 323

TITLE: Regular Spherical Bessel Functions

TYPE: Closed

NUMBER OF WORDS: 159 (124 plus S3 and T5)

TEMPORARY STORAGE: 0-7 plus N+1 locations at S3

PRESET PARAMETER: S3 contains the address where the table of functions $j_0(x), j_1(x), \ldots, j_N(x)$ is to be placed.

DURATION: Variable (approx. 200 ms.) depends on N and the relationship between x and N.

ACCURACY: Maximum error $2 \cdot 10^{-11}$. Usually between 10^{-11} and 10^{-12}.

DESCRIPTION: Enter with scaled argument $2^{-5}x$ in the accumulator and the program parameter:

p	50 N F
	50 pF
p + 1	26 qF

$$\frac{1}{16} \leq x \leq 19$$
$$N \leq 31$$

where q is the location of this program and N is the maximum order of the Bessel Function to be found. N should not be less than 3. If x is greater than N + 1/2, N will be changed to the next integer larger than x. On exit $j_k(x)$ will be found in kS3.

NOTE: This program contains Logrithm routine S3 at 124L and Sine-Cosine routine T5 at 138L. These are available for use in other connections. Merely enter at 124L for S3 and 138L for T5.

METHOD: A complete discussion of the method may be found in the "Journal of the Association for Computing Machinery" Vol. 6, page 366. (July, 1959).

This ppogram starts by choosing a k greater than N and assumes that $\bar{j}_k = 0$ and $\bar{j}_{k-1}$ is arbitrary. It then computes the ratios $\bar{r}_n = \frac{\bar{j}_{n+1}}{\bar{j}_n}$ by using the

recursion relation $\bar{r}_{n-1} = \dfrac{x}{2n + 1 - x\bar{r}_n}$; $\bar{r}_k = 0$.

When r_n becomes greater than one, the program starts to use the recursion relation for j's:

$$\bar{j}_{n-1} = \frac{2n + 1}{x} \bar{j}_n - \bar{j}_{n+1}$$

The $\bar{j}$'s so found will be proportional to the true j's. the proportionality constant α is

$$\alpha = \frac{\bar{j}_n}{j_n} = (\bar{j}_0 - x\bar{j}_1) \cos x + x\bar{j}_0 \sin x.$$

The true j's are then found using α and the ratios found before.

The ratio r_n becomes greater than one when the function leaves the monotonic region and begins to oscillate. Therefore r_n is not used in the oscillating region since r_n could blow up at the zeros of the function.

One should note that $\bar{r}_n = r_n$.

ERRORS: This method is not subject to the large errors to which the simpler method of upward recursion from j_0 and j_1 is subject. Since the ratios are always of the order of unity, one retains full accuracy. The $\bar{j}$'s and j's can be made as proportional as one wishes by choosing k to be sufficiently high.

This program was checked using existing tables of the functions. For x less than 10, the errors were about 10^{-11}. For x greater than 10, this program has greater accuracy than the tables.

DATE June 9, 1961
PROGRAMMED BY R. Parsons
APPROVED BY K W Dickman

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ON THE FOLLOWING FIVE PAGES ARE TABULATED REPRESENTATIVE VALUES FOR THE REGULAR SPHERICAL BESSEL FUNCTIONS. THE LEFT COLUMN GIVES VALUES COMPUTED BY ILLIAC. THE RIGHT COLUMN ARE THE VALUES GIVEN IN TABLES OF SPHERICAL BESSEL FUNCTIONS (VOLUME 1). IN NO CASE IS THE ERROR GREATER THAN $2 \cdot 10^{-11}$. IN ALL CASES WHERE THE TABLES DO NOT GIVE THE VALUE TO ELEVEN FIGURES, THE ILLIAC VALUE ROUNDED IS THE VALUE GIVEN IN THE TABLE.

THE VALUES ARE TABULATED FOR A GIVEN x IN THE FOLLOWING ORDER: $j_0(x)$, $j_1(x)$, ..., $j_{15}(x)$. VOLUME 1 OF THE TABLES DOES NOT LIST ORDERS HIGHER THAN 13.

x = .1

ILLIAC	TABLES
+99833416647**6**	+9983341665
+033300011894	+03330001190
+000666190606	+000666190606
+000009518519	+0000095185197
+000000105771	+000000105772
+000000000960	+000000000961
+000000000007	+000000000074
+000000000000	+000000000000
+000000000000	+000000000000
+000000000000	+000000000000
+000000000000	+000000000000
+000000000000	+000000000000
+000000000000	+000000000000
+000000000000	+000000000000
+000000000000	

x = .2

ILLIAC	TABLES
+993346654015	+9933466540
+066400380676	+06640038067
+002659056079	+002659056080
+000076021317	+000076021318
+000001690045	+000001690046
+000000030735	+000000030736
+000000000473	+000000000473
+000000000005	+000000000006
+000000000000	+000000000000
+000000000000	+000000000000
+000000000000	+000000000000
+000000000000	+000000000000
+000000000000	+000000000000
+000000000000	+000000000000
+000000000000	

x = .5

ILLIAC	TABLES
+958851077232	+9588510772
+162537030634	+1625370306
+016371106605	+01637110661
+001174035442	+00117 4035444
+000065389606	+000065389606
+000002977466	+000002977467
+000000114664	+000000114665
+000000003825	+000000003826
+000000000111	+000000000112
+000000000002	+000000000003
+000000000000	+000000000000
+000000000000	+000000000000
+000000000000	+000000000000
+000000000000	
+000000000000	
+000000000000	

x = 1.0

ILLIAC	TABLES
+8414709848886	+8414709848
+301168678945	+3011686789
+062035052009	+06203505201
+009006581116	+00900658112
+001011015807	+001011015808
+000092561158	+000092561158
+000007156936	+000007156936
+000000479013	+000000479013
+000000028263	+000000028265
+000000001490	+000000001491
+000000000071	+000000000071
+000000000002	+000000000003
+000000000000	+000000000000
+000000000000	+000000000000
+000000000000	
+000000000000	

x = 3.0

ILLIAC	TABLES
+0470400027	+0470400027
+3456774998	+3456774998
+2986374971	+2986374971
+1520516620	+1520516620
+0561497143	+0561497143
+0163974809	+0163974809
+0039743825	+0039743825
+0008248432	+0008248432
+0001498337	+0001498337
+0000242146	+0000242146
+0000035260	+0000035260
+0000004673	+0000004673
+0000000568	+0000000568
+0000000063	+0000000063

x = 5.0

ILLIAC	TABLES
-1917848549	-1917848549
-0950894081	-0950894081
+1347312101	+1347312101
+2298206182	+2298206182
+1870176553	+1870176553
+1068111614	+1068111614
+0479668998	+0479668998
+0179027781	+0179027781
+0057414346	+0057414346
+0016180997	+0016180997
+0004073442	+0004073442
+0000927461	+0000927461
+0000192878	+0000192878
+0000036932	+0000036932
+0000006554	+0000006554
+0000001084	+0000001084

x = 10.0

ILLIAC	TABLES
-0544021111078	-054402111109
+078466941812	+07846694
+077942193615	+07794219363
-039495844992	-03949584
-105589285105	-1055892851
-055534511603	-05553451162
+044501322349	+04450132233
+113386230658	+11338623
+125578023653	+125578023 6
+100096409551	+10009640955
+064605154494	+06460515449
+035574414886	+03557441
+017215999744	+017216000
+007465584475	+007465584
+002941078841	
+001063542713	

x = 15.0

ILLIAC	TABLES
+043352522667	+04335252
+053536029051	+05353603
-032645316858	-03264532
-064417801330	-06441780
+002583676234	+00258368
+065968007077	+06596801
+045792862291	+04579286
-026280859767	-02628086
-072073722049	-07207372
-055402691882	-05540269
+001896979005	+00189698
+058058462489	+05805846
+087125996813	+08712600
+087151532201	+08715153
+069746761150	
+047692206019	

x = 19.0

ILLIAC	TABLES
+007888274196	+00788827
-051621912842	-05162191
-016039102537	-01603910
+047401096386	+04740110
+033502664361	+03350266
-031531413262	-03153141
-051757693096	-05175769
-003881745173	-00388175
+048693157436	+04869316
+047449307092	+04744931
-001243850345	-00124385
-048824089050	-04882409
-057858994291	-05785899
-027306166596	-02730617
+019055494386	
+056390868553	

LOCATION	ORDER	NOTES	PAGE 1	V 14
	00K(V14)			
0	40 3F	$2^{-5}x \longrightarrow 3$		
	41 4F			
1	K5 F	Plant link		
	42 110L			
2	10 20F			
	42 4F	$N \cdot 2^{-39} \longrightarrow 4$		
3	50 111L	$q = 0$		
	L5 4F			
4	00 1F			
	40 F	$N(0) = n \cdot 2^{-38}$		
5	L5 3F			
	10 33F			
6	F0 F	$x - (n + 1/2) \cdot 2^{-38}$		
	32 7L	≥ 0		
7	22 9L	< 0		
	L5 3F			
8	10 34F			
	F4 111L	$n = [x] + 1$		
9	40 4F			
	50 111L			
10	L5 4F			
	00 1F			
11	F4 111L			
	00 33F	$(N + 1/2) \cdot 2^{-5}$		
12	40 F			
	L5 3F			
13	66 F			
	S5 S3			
14	40 F	$N(0) = u^1 = \frac{x}{n + 1/2}$		
	50 F			
15	7J F			
	40 1F	$N(1) = u^{1^2}$		
16	10 1F			
	L4 112L			

LOCATION	ORDER	NOTES PAGE 2 V 14
17	40 2F	$u^{1^2}/2 - 1$
	50 2F	
18	7J F	
	40 F	
19	50 F	
	7J 113L	
20	10 4F	
	40 F	$N(0) = .35u^1(1 - u^{1^2}/2) \cdot 2^{-4}$
21	L5 1F	
	L4 112L	
22	40 1F	$N(1) = (-1 + u^{1^2})$
	50 111L	
23	L5 F	
	66 1F	
24	S5 F	
	L4 114L	
25	10 16F	
	50 25L	
26	26 124L	
	40 F	
27	L1 F	
	F4 111L	
28	L4 4F	ν
	50 111L	$q = 0$
29	10 7F	
	S5 F	
30	40 F	$N(0) = \nu \cdot 2^{-7}(n)$
	50 111L	
31	L5 4F	
	10 7F	
32	S5 F	
	40 1F	$N(1) = N \cdot 2^{-7}$
33	L5 13L	
	L4 4F	
34	42 43L	Initialize store
	41 2F	$r = 0$ N(2)

LOCATION	ORDER	NOTES PAGE 3 V 14
35	50 3F	
	79 2F	
36	L4 115L	
	L4 F	
37	L4 F	$(2\ n+1 - xr_n) \cdot 2^{-7}$
	40 2F	temp. store
38	50 111L	
	L5 3F	
39	10 4F	
	66 2F	
40	S5 F	
	40 2F	r_{n-1}
41	L5 1F	
	L0 F	$(N-n) \cdot 2^{-7}$
42	36 43L	
	22 45L	
43	L5 2F	
	40 F	store r_n in (n+1) S3
44	L5 43L	
	L0 116L	
45	40 43L	
	L5 F	
46	L0 115L	n = n-1
	40 F	
47	L0 117L	$n \leq 1$?
	36 51L	yes
48	50 111L	no
	L5 2F	
49	00 1F	
	40 5F	temp. store
50	LL 5F	
	36 35L	if $r_n < 1$
51	L5 43L	
	42 52L	

LOCATION	ORDER	NOTES PAGE 4 V 14
52	L5 118L	
	40 F	$j_n = 2^{-6}$
53	L5 F	
	40 5F	$N(5) = n \cdot 2^{-7}$ at transfer time
54	50 111L	
	00 2F	
55	40 F	$N(0) = n \cdot 2^{-5}$
	L5 118L	
56	40 1F	$N(1) = j_n$ (2^{-6}) initially
	L5 2F	
57	10 4F	initially
	40 2F	$N(2) = j_{n+1}$ $(2^{-6} r_n)$
58	L5 52L	
	L0 116L	
59	42 64L	Set store address
	L5 F	
60	L4 F	
	L4 119L	$(2n + 1) \cdot 2^{-5}$
61	40 6F	temp. store
	50 6F	
62	75 1F	
	66 3F	$(\frac{2n+1}{x} \ \bar{j}_n) \cdot 2^{-6}$
63	S5 F	
	L0 2F	
64	40 6F	temp. store
	40 F	(by 59)
65	L5 1F	
	40 2F	$j_{n+1} = \bar{j}_n$
66	L5 6F	
	40 1F	$j_n = j_{n-1}$
67	L5 64L	
	L0 116L	
68	40 64L	decrease store address
	L5 F	
69	L0 119L	
	40 F	n:n-1

LOCATION	ORDER	NOTES PAGE 5 V 14
70	L0 119L	$n \leq 0$
	32 59L	no
71	L5 S3	yes
	10 1F	
72	40 F	$\bar{j}_0 \times 2^{-1} \longrightarrow 0$
	50 1S3	
73	71 3F	
	00 4F	
74	L4 F	$(\bar{j}_0 - x_{\bar{j}_1}) 2^{-1}$
	40 6F	$N(6) \nearrow^{\bar{j}_1}$
75	50 120L	
	75 3F	
76	00 5F	
	40 7F	$N(7) = x/\pi \mod 2$
77	LJ 7F	
	50 77L	
78	26 138L	
	40 F	
79	50 F	
	7J 6F	$(\bar{j}_0 - x_{\bar{j}_1}) \cos x \cdot 2^{-2}$
80	40 6F	
	L5 7F	
81	40 F	waste
	50 81L	
82	26 138L	
	40 F	
83	50 F	
	7J S3	$(\sin x)\, j_0 \cdot 2^{-1}$
84	40 F	
	50 F	
85	75 3F	
	00 4F	$(\sin x)\, \bar{j}_0 \times \cdot 2^{-2}$
86	L4 6F	
	40 1F	$N(1) = \alpha \cdot 2^{-2}$
87	L5 4F	N
	L4 121L	

LOCATION	ORDER	NOTES PAGE 6 V 14
88	42 122L	
	42 123L	
89	L5 5F	
	10 32F	
90	L4 121L	
	40 5F	end constant
91	L5 83L	
	42 93L	
92	42 95L	
	40 F	waste
93	50 111L	
	L5 F	
94	10 2F	$\overline{j}_k \cdot 2^{-2}$
	66 1F	
95	S5 F	
	40 F	store j_n
96	F5 93L	
	40 93L	
97	F5 95L	
	40 95L	
98	L0 122L	
	32 110L	end
99	L5 95L	
	L0 5F	
100	36 93L	
	L5 95L	
101	42 105L	
	42 104L	
102	L0 116L	
	42 103L	
103	40 F	waste
	50 F	
104	40 F	waste
	75 F	$r_n \, j_{n-1} = j_n$
105	00 2F	
	40 F	

LOCATION	ORDER	NOTES
106	F5 103L	
	40 103L	
107	F5 104L	increment address
	40 104L	
108	F5 105L	
	40 105L	
109	L0 123L	
	32 103L	
110	40 F	waste
	22 F	exit
111	00 F	0
	00 F	
112	80 F	-1
	00 F	
113	00 F	
	00 3500 0000 0000J	
114	00 F	
	00 62 5000 0000J	
115	01 F	
	00 F	2^{-7}
116	00 F	2^{-39}
	00 1F	
117	82 F	
	00 F	$(-)\ 2 \cdot 2^{-7}$
118	02 F	
	00 F	2^{-6}
119	04 F	
	00 F	2^{-5}
120	00 F	
	00 3183 0988 6184J	$1/\pi$
121	35 F	
	40 1S3	
122	S5 F	
	40 F	

LOCATION	ORDER	NOTES PAGE 8 V 14
123	80 2F	
	40 F	
124	(S3) 00K	
138	(T5)00K	